Hence

$$\mathcal{H}^n_\delta([0,1]^n) \geq d_n \cdot 2^{-n} > 0$$

$$\Rightarrow \mathcal{H}^n([0,1]^n) > 0. \qquad \blacksquare$$

§ 3.5. Hausdorff dimension.

Def. Let $A \subset \mathbb{R}^n$. Define

$$\dim_H A := \sup\{s \geq 0 : \mathcal{H}^s(A) > 0\}$$
$$= \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\}$$

We call it the Hausdorff dimension.

Facts: (1) If $A \subset B$, then $\dim_H A \leq \dim_H B$

(2) If $A = \bigcup_{i=1}^{\infty} A_i$ with A, A_i being Borel, then

$$\dim_H A = \sup_i \dim_H A_i$$

Def. A function $f: A \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be Hölder continuous with exponent α if $\exists\ M > 0$ such that

$$(***) \quad |f(x) - f(y)| \leq M \cdot |x-y|^\alpha,$$

for all $x, y \in A$.

If $\alpha = 1$, then we call f is Lipschitz continuous.

Prop 3.11. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^n$ be Hölder cts with exponent α, and const M as in $(***)$.

Then for $s \geq 0$,

$$\mathcal{H}^{s/\alpha}(f(A)) \leq M^{s/\alpha} \cdot \mathcal{H}^s(A).$$

As a consequence, $\dim_H f(A) \leq \dim_H A / \alpha$.

Let $s \geq 0$.

Pf. Let $\delta > 0$. Let $\varepsilon > 0$.

Pick a δ-cover $\{C_j\}$ of A such that

$$\sum_j |C_j|^s \leq \mathcal{H}^s_\delta(A) + \varepsilon$$

Then $|f(C_j)| \leq M \cdot |C_j|^\alpha$ by the Hölder cty assumption on f.

Hence

$$\sum_j |f(C_j)|^{s/\alpha} \leq \sum_j M^{s/\alpha} \cdot \left(|C_j|^\alpha\right)^{s/\alpha}$$

$$= \sum_j M^{s/\alpha} \cdot |C_j|^s$$

$$\leq M^{s/\alpha} \cdot \left(\mathcal{H}^s_\delta(A) + \varepsilon\right)$$

But $\{f(C_j)\}_{j=1}^\infty$ is a $M \cdot \delta^\alpha$- cover of $f(A)$.

Hence

$$\mathcal{H}^{s/\alpha}_{M\cdot\delta^\alpha}(f(A)) \le M^{s/\alpha}\cdot\left(\mathcal{H}^s_\delta(A)+\varepsilon\right).$$

Letting $\delta\to 0$, then letting $\varepsilon\to 0$, gives

$$\mathcal{H}^{s/\alpha}(f(A)) \le M^{s/\alpha}\,\mathcal{H}^s(A).$$

$\blacksquare$

Example 1: Let $f:[0,1]\to\mathbb{R}$ be a Lipschitz function with Lip constant M. That is,

$$|f(x)-f(y)| \le M|x-y|, \quad \forall\, x,y\in[0,1].$$

Let $G_f = \{(x,f(x)) : x\in[0,1]\}\subset\mathbb{R}^2$.

Then

$$1\le \mathcal{H}^1(G_f)\le\sqrt{M^2+1}.$$

so $\dim_H G_f = 1$.

pf. Define $g : [0,1] \to G_f \subset \mathbb{R}^2$ by

$$x \mapsto (x, f(x)).$$

Then

$$|g(x) - g(y)| = \sqrt{(x-y)^2 + (f(x) - f(y))^2}$$

$$\leq \sqrt{M^2+1}\, |x-y|.$$

Applying Prop 3.11,

$$\mathcal{H}^{1/1}(g([0,1])) \leq \left(\sqrt{M^2+1}\right)^{1/1} \cdot \mathcal{H}^1([0,1])$$

$$\Rightarrow \mathcal{H}^1(G_f) \leq \sqrt{M^2+1}.$$

To see the other direction, notice that

$$g^{-1} : G_f \to [0,1], \quad (x, f(x)) \mapsto x.$$

Then $|g^{-1}(u) - g^{-1}(v)| \leq |u-v|$, $\forall\, u, v \in G_f$

(check it!)

Hence by Prop 3.11, (letting $s=1$, $\alpha=1$)

$$\mathcal{H}^{1/1}\left(g^{-1}(G_f)\right) \le 1 \cdot \mathcal{H}^1(G_f)$$

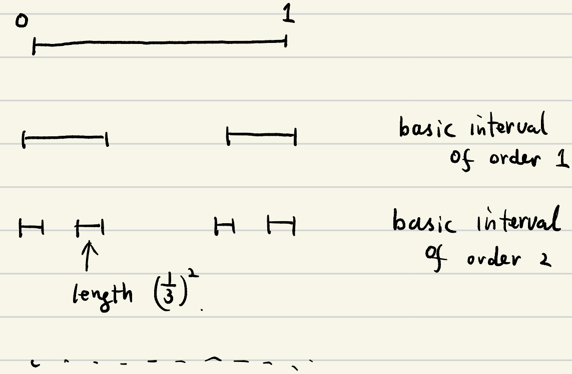
That is, $1 = \mathcal{H}^1([0,1]) \le \mathcal{H}^1(G_f)$.

$\boxed{/\!/}$

Example 2. Let C be the middle-third Cantor set. Find $\dim_H C$.

Solution:

Construction of C.



From the construction of C, we see that for any $n \in \mathbb{N}$, C can be covered by 2^n many basic intervals of order n. Each such interval has length 3^{-n}.

Hence
$$\mathcal{H}^s_{3^{-n}}(C) \le 2^n \cdot (3^{-n})^s = 2^n \cdot 3^{-ns}$$

$$= 2^{n\left(1 - s\cdot(\log 3/\log 2)\right)},$$

Letting $s = \log 2/\log 3$ gives

$$\mathcal{H}^{\log 2/\log 3}_{3^{-n}}(C) \leq 1.$$

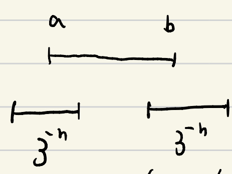
$$\Rightarrow \mathcal{H}^{\log 2/\log 3}(C) \leq 1.$$

We claim that $\mathcal{H}^{\log 2/\log 3}(C) > 0.$

(Outline): Let μ be the Cantor measure,

i.e. μ is a prob. measure supported on C such that $\mu(I) = 2^{-n}$ for any basic interval I of order n.

Notice that any interval $[a,b]$ with lengthen between 3^{-n-1} and 3^{-n} intersects at most 2 basic intervals of order n.



Hence $\mu([a,b]) \leq 2\cdot 2^{-n}. = 2\ 3^{-(\log 2/\log 3)n}$

$$= 2\cdot\left(3^{-n}\right)^{\log 2/\log 3}$$

$$\leq 2 \cdot \left(3(b-a)\right)^{\log 2/\log 3}$$

It follows that $\exists$ a constant $c > 0$ such that

$$\mu([a,b]) \leq d \cdot \underline{(b-a)}^{\log 2/\log 3}.$$

(****)

Let $\{A_i\}$ be a δ-cover of C.

Let $a_i = \inf A_i$, $b_i = \sup A_i$

Then $\{[a_i, b_i]\}$ is a δ-cover of C

with $\sum_i |A_i|^s = \sum_i |b_i - a_i|^s$

Let $s = \log 2/\log 3$. Then

$$\sum_i |A_i|^s = \sum_i |b_i - a_i|^s$$

$$\geq \frac{1}{d} \sum_i \mu([a_i, b_i]) \quad (\text{using } ****)$$

$$\geq \frac{1}{d} \mu(C) \geq \frac{1}{d}.$$

Hence $\mathcal{H}^s_\delta(C) \geq \frac{1}{d}$, $\forall\ \delta > 0$

Letting $\delta \to 0$ gives $\mathcal{H}^s(C) \geq \frac{1}{d} > 0$.